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1976 J. Phys. A: Math. Gen. 9 863

(http://iopscience.iop.org/0305-4470/9/6/008)

J. Phys. A: Math. Gen., Vol. 9, No. 6, 1976. Printed in Great Britain. © 1976

The evaluation of weight multiplicities using characters and S-functions

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Received 21 January 1976

Abstract. Weight vectors and weight multiplicities are defined in terms of group characters. The characters appropriate to all the unitary irreducible representations of the unitary, orthogonal and symplectic groups are expressed, following techniques introduced by Littlewood, in terms of S-functions. The resulting explicit formulae for weight multiplicities are used to tabulate results by making use of the definition of S-functions in terms of standard Young tableaux. The results obtained give, for the first time, the k dependence of the weight multiplicities of the groups $U(k)$, $O(2k+1)$, $Sp(2k)$ and $O(2k)$. There is no limitation on the size of k nor on the dimensions of the representations.

1. Introduction

The multiplicities of the weights associated with the basis states of an irreducible unitary representation of any compact semi-simple Lie group may be calculated by a variety of methods. For example use may be made of the character formula derived by Weyl (1926), the explicit formula for the weight multiplicity due to Kostant (1959), or the recurrence relations due to Freudenthal (1964) and Racah (1964).

The most efficient algorithms developed have been based on Freudenthal's recurrence relation. For example computer implementations of this relation have been made by Agrawala and Belinfante (1969), Krusemeyer (1971), Beck and Kolman (1972a) and Kolman and Beck (1973a). No difficulties are experienced with groups of rank as large as 9 for representations of dimension not greater than 1000.

Despite this success all such calculations are rank dependent. This is particularly unfortunate in the case of the unitary groups for which the weight multiplicities are in fact independent of rank. In the case of the orthogonal and symplectic groups this is no longer true but it is still disappointing that these calculations give no clue as to the nature of the dependence of the weight multiplicities on the rank.

This situation may be remedied by making use of S-function techniques to derive new algorithms for determining weight multiplicities for all the classical Lie groups. This possibility arises because weights and their multiplicities may be defined in terms of characters. Indeed the characters $\chi_{\boldsymbol{\phi}}^{\lambda_G}$ of an irreducible representation $\boldsymbol{\lambda}_G$ of a compact semi-simple Lie group G of rank k may be written in the form:

$$\chi_{\boldsymbol{\phi}}^{\lambda_G}=\sum_{\boldsymbol{m}} M_{\boldsymbol{m}}^{\lambda_G}\, e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}} \tag{1.1}$$

where $\boldsymbol{\phi}=(\phi_1, \phi_2, \ldots, \phi_k)$ is a set of k class parameters. If the coefficient $M_{\boldsymbol{m}}^{\lambda_G}$ is non-vanishing then $\boldsymbol{m}=(m_1, m_2, \ldots, m_k)$ is said to be a weight vector of the representation $\boldsymbol{\lambda}_G$ having multiplicity $M_{\boldsymbol{m}}^{\lambda_G}$.

In the case of the classical Lie groups the characters may be expressed in terms of characters of irreducible representations of the unitary groups and these latter characters may be defined in terms of the particular symmetric functions known as S-functions. Finally S-functions themselves have a simple combinatorial definition in terms of standard Young tableaux which allows them to be calculated in a very straightforward manner.

The definition of the k class parameters $(\phi_1, \phi_2, \ldots, \phi_k)$ in terms of the maximal toroidal subgroup, T_G, of G is given in § 2 along with the explicit definition of T_G in the cases for which G is any one of the groups $\mathrm{U}(k)$, $\mathrm{O}(2k+1)$, $\mathrm{Sp}(2k)$ and $\mathrm{O}(2k)$. This is followed in § 3 by the introduction of the notation appropriate to S-functions and their definition in terms of standard Young tableaux and the all important Kostka matrix which is the basis of the enumerations carried out here.

The irreducible representations of the classical Lie groups are specified in § 4. Through the use of S-function techniques involving certain infinite series developed by Littlewood (1950) character formulae are then derived. These are used in § 5 to evaluate explicitly weights and their multiplicities. Finally the results are tabulated and some comments made regarding the relationships between multiplicities for different groups.

2. Weights and their multiplicities

The weights and the corresponding weight multiplicities of an irreducible representation $\boldsymbol{\lambda}_G$ of a compact semi-simple Lie group G of rank k are defined through the expression (1.1) for the characters $\chi_{\boldsymbol{\phi}}^{\lambda_G}$ of such a representation in terms of a set of k parameters $\phi_1, \phi_2, \ldots, \phi_k$. These parameters may be defined through the isomorphism between the maximal toroidal subgroup T_G of G and the group

$$T_k=\mathrm{U}(1)\times \mathrm{U}(1)\times \ldots \times \mathrm{U}(1) \tag{2.1}$$

which is the direct product of k groups $\mathrm{U}(1)$.

The group elements of T_k are denoted by

$$(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_k}) \qquad \text{with } 0 \leqslant \phi_i < 2\pi$$

for $i=1,2,\ldots,k$. The classical Lie groups $\mathrm{U}(k)$, $\mathrm{O}(2k+1)$, $\mathrm{Sp}(2k)$ and $\mathrm{O}(2k)$ are each of rank k. The element of the corresponding maximal toroidal subgroup T_G which, under the isomorphism between T_G and T_k, maps onto the element $(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_k})$ is given for each of these groups G by:

$$\mathrm{U}(k) \qquad \mathrm{e}^{\mathrm{i}\phi_1}\dot{+}\mathrm{e}^{\mathrm{i}\phi_2}\dot{+}\ldots\dot{+}\mathrm{e}^{\mathrm{i}\phi_k} \tag{2.2}$$

$$\mathrm{O}(2k+1) \qquad \begin{bmatrix}\cos\phi_1 & -\sin\phi_1\\ \sin\phi_1 & \cos\phi_1\end{bmatrix}\dot{+}\begin{bmatrix}\cos\phi_2 & -\sin\phi_2\\ \sin\phi_2 & \cos\phi_2\end{bmatrix}\dot{+}\ldots\dot{+}\begin{bmatrix}\cos\phi_k & -\sin\phi_k\\ \sin\phi_k & \cos\phi_k\end{bmatrix}\dot{+}1 \tag{2.3}$$

$$\mathrm{Sp}(2k) \qquad \mathrm{e}^{-\mathrm{i}\phi_1}\dot{+}\mathrm{e}^{\mathrm{i}\phi_1}\dot{+}\mathrm{e}^{-\mathrm{i}\phi_2}\dot{+}\mathrm{e}^{\mathrm{i}\phi_2}\dot{+}\ldots\dot{+}\mathrm{e}^{-\mathrm{i}\phi_k}\dot{+}\mathrm{e}^{\mathrm{i}\phi_k} \tag{2.4}$$

$$\mathrm{O}(2k) \qquad \begin{bmatrix}\cos\phi_1 & -\sin\phi_1\\ \sin\phi_1 & \cos\phi_1\end{bmatrix}\dot{+}\begin{bmatrix}\cos\phi_2 & -\sin\phi_2\\ \sin\phi_2 & \cos\phi_2\end{bmatrix}\dot{+}\ldots\dot{+}\begin{bmatrix}\cos\phi_k & -\sin\phi_k\\ \sin\phi_k & \cos\phi_k\end{bmatrix}. \tag{2.5}$$

The importance of T_G lies in the fact that every element of G belonging to the component of G which is connected to the identity element is conjugate to an element of T_G. This has been proved explicitly for the classical Lie groups by, for example, Weyl (1946, pp 179, 217) and Littlewood (1950, pp 16, 18). It follows that if any group element of the class parametrized by $\boldsymbol{\phi}$ is denoted by the matrix A then the eigenvalues of such an element are:

$$\mathrm{U}(k) \qquad (\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_k}) \tag{2.6}$$

$$\mathrm{O}(2k+1) \qquad (\mathrm{e}^{-\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{-\mathrm{i}\phi_2}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{-\mathrm{i}\phi_k}, \mathrm{e}^{\mathrm{i}\phi_k}, 1) \tag{2.7}$$

$$\mathrm{Sp}(2k) \qquad (\mathrm{e}^{-\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{-\mathrm{i}\phi_2}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{-\mathrm{i}\phi_k}, \mathrm{e}^{\mathrm{i}\phi_k}) \tag{2.8}$$

$$\mathrm{O}(2k) \qquad (\mathrm{e}^{-\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{-\mathrm{i}\phi_2}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{-\mathrm{i}\phi_k}, \mathrm{e}^{\mathrm{i}\phi_k}) \tag{2.9}$$

as can be seen from (2.2)–(2.5).

In any representation $\boldsymbol{\lambda}_G$ of the group G the element A is mapped onto a matrix A^{λ_G} whose trace is the character:

$$\chi_{\boldsymbol{\phi}}^{\lambda_G} = \chi^{\lambda_G}(A) = \mathrm{Tr}\, A^{\lambda_G}. \tag{2.10}$$

It is clear from the definition of the parameters $\boldsymbol{\phi}$ for each of the classical Lie groups that these characters will be functions of $\boldsymbol{\phi}$ invariant under permutations of $\phi_1, \phi_2, \ldots, \phi_k$ and, in the case of the orthogonal and symplectic groups, invariant under arbitrarily distributed changes of sign of these parameters. This invariance is reflected in the symmetry of the weight diagrams constructed by assigning to each point in the Euclidean weight space specified by a weight vector $\boldsymbol{m} = (m_1, m_2, \ldots, m_k)$ the appropriate weight multiplicity $M_{\boldsymbol{m}}^{\lambda_G}$. The corresponding symmetry group, known as the Weyl group, is the symmetric group, S_k, for $\mathrm{U}(k)$ and the hyperoctohedral group, Q_k, for $\mathrm{O}(2k+1)$, $\mathrm{Sp}(2k)$ and $\mathrm{O}(2k)$.

3. S-functions

The symbol $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ is used to denote a partition of l into p non-vanishing parts satisfying the conditions $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_p = l$. It is sometimes convenient to write $\boldsymbol{\lambda} \vdash l$ to signify that $\boldsymbol{\lambda}$ is a partition of l. Such partitions may be used to label the irreducible representations of the symmetric group S_l. Corresponding to each representation there exists a particular symmetric function of a set of indeterminates $x_1, x_2, \ldots, x_n$ called by Littlewood (1950, p 84) an S-function. This S-function may conveniently be written as $\mathrm{e}_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_n)$ following the notation of Stanley (1971) who defined this function neither in terms of the immanants of a matrix, nor in terms of bi-alterants, but in terms of standard Young tableaux.

The Young diagram corresponding to the partition $\boldsymbol{\lambda}$ consists of l boxes arranged in rows of length $\lambda_1, \lambda_2, \ldots, \lambda_p$. A standard Young tableau is one in which numbers are inserted into each of the boxes of a Young diagram in such a way that the numbers are non-decreasing reading from left to right across each row and are strictly increasing reading from top to bottom down each column. With this terminology S-functions are defined by

$$\mathrm{e}_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_n) = \sum_{\boldsymbol{a}} K_{\boldsymbol{a}}^{\boldsymbol{\lambda}} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \tag{3.1}$$

where the summation is carried out over all vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ whose components are non-negative integers, and where the coefficient $K^\lambda_{\boldsymbol{a}}$ is the number of standard Young tableaux of shape defined by the partition $\boldsymbol{\lambda}$ containing the numbers $1, 2, \ldots, n$ precisely $a_1, a_2, \ldots, a_n$ times, respectively.

In any row, r, of such a standard Young tableau there will appear a succession of f_r entries i having no entry $i+1$ beneath them and g_r entries $i+1$ having no entry i above them. If this, possibly null, succession of $(f_r + g_r)$ entries is changed into a succession of g_r entries i followed by f_r entries $i+1$ the resulting tableau is still standard, as can be seen from figure 1. The application of this same change to the entries in each row,

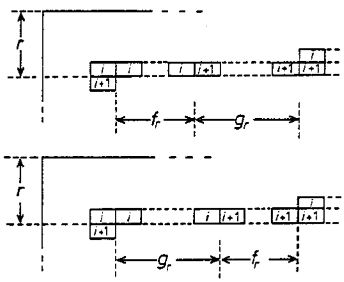

Figure 1.

$r = 1, 2, \ldots, p$, leads to the conclusion that any standard Young tableau corresponding to the vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_n)$ is changed into one of the same shape corresponding to the vector $\boldsymbol{a}' = (a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_n)$. Moreover this change provides a bijection between the standard Young tableaux associated with $\boldsymbol{a}$ and $\boldsymbol{a}'$, so that $K^\lambda_{\boldsymbol{a}'} = K^\lambda_{\boldsymbol{a}}$. Using the fact that the transpositions $(i, i+1)$ for $i = 1, 2, \ldots, n-1$ generate the symmetric group S_n it follows that

$$K^\lambda_{\pi(\boldsymbol{a})} = K^\lambda_{\boldsymbol{a}}, \tag{3.2}$$

where $\pi(\boldsymbol{a}) = (a_{\pi_1}, a_{\pi_2}, \ldots, a_{\pi_n})$ for any element π of S_n. It is then an immediate consequence of the definition (3.1) that each S function is a symmetric function of the indeterminates $(x_1, x_2, \ldots, x_n)$.

Monomial symmetric functions corresponding to a partition $\boldsymbol{\mu}$ of l into q non-vanishing parts $\mu_1, \mu_2, \ldots, \mu_q$ are defined by:

$$k_\mu(x_1, x_2, \ldots, x_n) = \sum_\sigma x_1^{\mu_{\sigma_1}} x_2^{\mu_{\sigma_2}} \ldots x_n^{\mu_{\sigma_n}} \tag{3.3}$$

where the summation is over those elements σ of S_n leading to distinct monomials with exponents in the order given by the permutation σ of the parts $\mu_1, \mu_2, \ldots, \mu_q$ and $(n-q)$ zeros. The completeness of the set of monomial symmetric functions and the fact that S-functions are themselves symmetric functions implies a linear relationship between them. It follows from (3.2) that this takes the form (Littlewood 1950, p 191)

$$e_\lambda(x_1, x_2, \ldots, x_n) = \sum_{\mu \vdash l} K^\lambda_\mu k_\mu(x_1, x_2, \ldots, x_n). \tag{3.4}$$

The coefficients K^λ_μ may be found, as special cases of the coefficients appearing in (3.1), through the enumeration of standard Young tableaux. However their values were first calculated and tabulated by Kostka (1882, 1907). The matrix, Kostka's matrix, whose elements are K^λ_μ, is non-singular as may be seen by the adoption of a lexicographical ordering for the partitions λ and μ labelling its rows and columns. With this choice the matrix is upper triangular with all diagonal elements unity. Its inverse, also determined by Kostka, is thus easy to find and its existence implies the completeness of S-functions as a set of symmetric functions.

Following Littlewood (1950, pp 94, 110) both products and quotients of S-functions may be defined. The product is simply:

$$e_{\lambda\cdot\mu}(x_1, x_2, \ldots, x_n) = e_\lambda(x_1, x_2, \ldots, x_n)e_\mu(x_1, x_2, \ldots, x_n) \tag{3.5}$$

and the quotient, as pointed out by Stanley (1971), is:

$$e_{\rho/\lambda}(x_1, x_2, \ldots, x_n) = \sum_a K_a^{\rho/\lambda} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}, \tag{3.6}$$

where $K_a^{\rho/\lambda}$ is the number of standard skew Young tableaux of shape defined by ρ/λ containing the numbers $1, 2, \ldots, n$ precisely $a_1, a_2, \ldots, a_n$ times. The skew Young diagram of shape ρ/λ is the diagram introduced by Robinson (1961, p 48) obtained from that corresponding to the partition ρ by the removal of the boxes corresponding to the partition λ.

The product (3.5) is obviously a symmetric function, and the same is true of the quotient (3.6) as may be seen by making use of the argument leading to (3.2), which now yields:

$$K_{\pi(a)}^{\rho/\lambda} = K_a^{\rho/\lambda}. \tag{3.7}$$

The expansions of these symmetric functions (3.5) and (3.6) in terms of S-functions take the forms:

$$e_{\lambda\cdot\mu}(x_1, x_2, \ldots, x_n) = \sum_{\rho\vdash l+m} g_{\lambda\mu}^{\rho} e_\rho(x_1, x_2, \ldots, x_n), \tag{3.8}$$

$$e_{\rho/\lambda}(x_1, x_2, \ldots, x_n) = \sum_{\mu\vdash r-l} g_{\lambda\mu}^{\rho} e_\mu(x_1, x_2, \ldots, x_n). \tag{3.9}$$

These are remarkable for the fact that the coefficients $g_{\lambda\mu}^{\rho}$ appearing in these expansions are identical. From (3.6), (3.9) and (3.1) it is clear that:

$$K_a^{\rho/\lambda} = \sum_\mu g_{\lambda\mu}^{\rho} K_a^{\mu}. \tag{3.10}$$

The rules for the evaluation of the coefficients $g_{\lambda\mu}^{\rho}$ have been given by Littlewood (1950, p 94). One important result which follows from a special case of these rules and the definition (3.1) is that

$$e_\lambda(x_1, x_2, \ldots, x_{n-1}, 1) = \sum_m e_{\lambda/m}(x_1, x_2, \ldots, x_{n-1}) \tag{3.11}$$

where the summation is carried out over all one-part partitions m. It is also to be noted that the rules imply that the product of q S-functions defined by the one-part partitions $\mu_1, \mu_2, \ldots, \mu_q$ is given by:

$$e_{\mu_1\cdot\mu_2\cdot\ldots\cdot\mu_q}(x_1, x_2, \ldots, x_n) = \sum_{\lambda\vdash m} K_\mu^{\lambda} e_\lambda(x_1, x_2, \ldots, x_n) \tag{3.12}$$

where the coefficients appearing here are the elements of Kostka's matrix which appeared earlier in (3.4).

4. Irreducible representations of the classical groups

The notation used here for the specification of the unitary irreducible representations of the classical Lie groups has been introduced elsewhere (King 1975). The ordinary irreducible representations of U(n), O(n) and Sp(n) corresponding to covariant tensors are denoted by $\{\lambda\}$, $[\lambda]$ and $\langle\lambda\rangle$ respectively. The composite irreducible representations of U(n) and O(n) corresponding to mixed tensors and to spinors are denoted by $\{\bar{\mu};\lambda\}$ and $[\Delta;\lambda]$ respectively.

The relationship between mixed tensor representations of U(n) and tensor representations which are either purely covariant or purely contravariant is such that:

$$\mathrm{U}(n) \qquad \{\bar{\mu};\lambda\}=\sum_{\zeta\vdash z}(-1)^z\{\overline{\mu/\zeta}\}\times\{\lambda/\tilde{\zeta}\} \tag{4.1}$$

$$\{\bar{\mu}\}\times\{\lambda\}=\sum_{\zeta\vdash z}\{\overline{\mu/\zeta};\lambda/\zeta\} \tag{4.2}$$

where the summations are over all partitions ζ, $\tilde{\zeta}$ is the partition conjugate to ζ, the symbol $\times$ indicates a Kronecker product and a superscript $-$ signifies the contragredient of a representation defined by taking the transpose of the inverse of the representation matrices. Similarly spinor representations are related to the basic spin representation $[\Delta]$, and tensor representations in such a way that:

$$\mathrm{O}(n) \qquad [\Delta;\lambda]=\sum_m(-1)^m[\Delta]\times[\lambda/m] \tag{4.3}$$

$$[\Delta]\times[\lambda]=\sum_m[\Delta;\lambda/\tilde{m}] \tag{4.4}$$

where the summations are carried out over all one-part partitions m.

Furthermore the representations of the classical groups are linked by the inverse restrictions (Littlewood 1950, pp 240, 295):

$$\mathrm{O}(n)\uparrow_{\mathrm{r}}\mathrm{U}(n) \qquad [\lambda]\uparrow_{\mathrm{r}}\sum_{\gamma\vdash c}(-1)^{c/2}\{\lambda/\gamma\}, \tag{4.5}$$

$$\mathrm{Sp}(n)\uparrow_{\mathrm{r}}\mathrm{U}(n) \qquad \langle\lambda\rangle\uparrow_{\mathrm{r}}\sum_{\alpha\vdash a}(-1)^{a/2}\{\lambda/\alpha\}, \tag{4.6}$$

where α and γ are specific types of partitions introduced by Littlewood (1950, p 238). In addition, two restrictions of interest take the form (King 1975):

$$\mathrm{O}(n)\downarrow\mathrm{O}(n-1) \qquad [\lambda]\downarrow\sum_m[\lambda/m] \tag{4.7}$$

$$[\Delta;\lambda]\downarrow\sum_m[\Delta;\lambda/m] \tag{4.8}$$

and

$$\sum_{m,\gamma\vdash c}(-1)^{m+c/2}\{\lambda/m\gamma\}=\sum_{\epsilon\vdash e}(-1)^{(e+r)/2}\{\lambda/\epsilon\}, \tag{4.9}$$

$$\sum_{m,\gamma\vdash c}(-1)^{c/2}\{\lambda/m\gamma\}=\sum_{\epsilon\vdash e}(-1)^{(e-r)/2}\{\lambda/\epsilon\}, \tag{4.10}$$

$$\sum_{m,\epsilon\vdash e}(-1)^{(e+r)/2}\{\lambda/m\epsilon\}=\sum_{\alpha\vdash a}(-1)^{a/2}\{\lambda/\alpha\} \tag{4.11}$$

where ϵ is a self-conjugate partition of e, of Frobenius rank r.

In terms of characters of irreducible representations of these groups the relationships (4.1), (4.3), (4.5), (4.6) and (4.9) imply that for any group element A

U(n) $$\chi^{\{\bar{\mu};\lambda\}}(A)=\sum_{\zeta\vdash z}(-1)^z\chi^{\{\mu/\zeta\}}(A)^*\chi^{\{\lambda/\tilde{\zeta}\}}(A), \tag{4.12}$$

O(n) $$\chi^{[\lambda]}(A)=\sum_{\gamma\vdash c}(-1)^{c/2}\chi^{\{\lambda/\gamma\}}(A), \tag{4.13}$$

$$\chi^{[\Delta;\lambda]}(A)=\chi^{\Delta}(A)\sum_{\epsilon\vdash e}(-1)^{(e+r)/2}\chi^{\{\lambda/\epsilon\}}(A), \tag{4.14}$$

Sp(n) $$\chi^{\langle\lambda\rangle}(A)=\sum_{\alpha\vdash a}(-1)^{a/2}\chi^{\{\lambda/\alpha\}}(A) \tag{4.15}$$

where use has been made of the fact that, for a unitary representation $\{\rho\}$, $\chi^{\{\bar{\rho}\}}(A)=\chi^{\{\rho\}}(A)^*$ with the asterisk signifying complex conjugation.

Littlewood (1950, p 188) showed that if A is an $n\times n$ matrix with eigenvalues $x_1, x_2, \ldots, x_n$ then the S-function $e_\lambda(x_1, x_2, \ldots, x_n)$ is the trace of the invariant matrix A^λ whose elements are polynomials in the elements of A. Furthermore if A is an element of a matrix group the map onto A^λ gives a representation of that group whose character is given by

$$\chi^\lambda(A)=e_\lambda(x_1, x_2, \ldots, x_n). \tag{4.16}$$

In addition Littlewood (1950, p 222) proved that in the case of the unitary group this representation is precisely the irreducible representation $\{\lambda\}$, so that

$$\chi^{\{\lambda\}}(A)=\chi^\lambda(A) \tag{4.17}$$

for all unitary matrices A including those which are the elements of the toroidal subgroups given by (2.2)–(2.5). Since in addition the spin characters of the elements (2.3) and (2.5) are known (Littlewood 1950, p 254) to be

$$\chi^\Delta(A)=\sum_{(\pm)}\exp[\tfrac{1}{2}\mathrm{i}(\pm\phi_1\pm\phi_2\pm\ldots\pm\phi_k)], \tag{4.18}$$

where the summation is carried over all possible $\pm$ sign combinations in the exponent, it is a straightforward matter to express the characters of all of the irreducible representations of the classical groups as explicit functions of $\boldsymbol{\phi}=(\phi_1, \phi_2, \ldots, \phi_k)$.

Making use, where necessary, of (3.9) and (4.7)–(4.11) the results are found to be

U(k) $$\chi^{\{\lambda\}}(A)=e_\lambda(\boldsymbol{\phi}) \tag{4.19}$$

$$\chi^{\{\bar{\mu};\lambda\}}(A)=\sum_{\zeta\vdash z}(-1)^z e_{\mu/\zeta}(-\boldsymbol{\phi})e_{\lambda/\zeta}(\boldsymbol{\phi}) \tag{4.20}$$

O($2k+1$) $$\chi^{[\lambda]}(A)=\sum_{\epsilon\vdash e}(-1)^{(e-r)/2}e_{\lambda/\epsilon}(\boldsymbol{\phi}, -\boldsymbol{\phi}) \tag{4.21}$$

$$\chi^{[\Delta;\lambda]}(A)=\sum_{\alpha\vdash a}(-1)^{a/2}e_\Delta(\boldsymbol{\phi})e_{\lambda/\alpha}(\boldsymbol{\phi}, -\boldsymbol{\phi}) \tag{4.22}$$

Sp($2k$) $$\chi^{\langle\lambda\rangle}(A)=\sum_{\alpha\vdash a}(-1)^{a/2}e_{\lambda/\alpha}(\boldsymbol{\phi}, -\boldsymbol{\phi}) \tag{4.23}$$

O($2k$) $$\chi^{[\lambda]}(A)=\sum_{\gamma\vdash c}(-1)^{c/2}e_{\lambda/\gamma}(\boldsymbol{\phi}, -\boldsymbol{\phi}) \tag{4.24}$$

$$\chi^{[\Delta;\lambda]}(A)=\sum_{\epsilon\vdash e}(-1)^{(e+r)/2}e_\Delta(\boldsymbol{\phi})e_{\lambda/\epsilon}(\boldsymbol{\phi}, -\boldsymbol{\phi}) \tag{4.25}$$

where it has been convenient to introduce the notation:

$$e_{\Delta}(\boldsymbol{\phi}) = \sum_{(\pm)} \exp[\tfrac{1}{2}\mathrm{i}(\pm\phi_1 \pm \phi_2 \ldots \pm \phi_k)] \tag{4.26}$$

$$e_{\lambda}(\boldsymbol{\phi}) = e_{\lambda}(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_k}) \tag{4.27}$$

$$e_{\nu}(\boldsymbol{\phi}, -\boldsymbol{\phi}) = e_{\nu}(\mathrm{e}^{\mathrm{i}\phi_1}, \mathrm{e}^{\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{\mathrm{i}\phi_k}, \mathrm{e}^{-\mathrm{i}\phi_1}, \mathrm{e}^{-\mathrm{i}\phi_2}, \ldots, \mathrm{e}^{-\mathrm{i}\phi_k}). \tag{4.28}$$

Almost all of these results were given first by Littlewood (1950) who went on to express the characters as quotients of determinants. However this is not appropriate for a comparison with the general formula (1.1). It is preferable to make use of the expansions (3.1) and (3.6) which imply that:

U(k)
$$\chi_{\boldsymbol{\phi}}^{\{\lambda\}} = \sum_{\boldsymbol{p}} K_{\boldsymbol{p}}^{\lambda}\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{\phi}} \tag{4.29}$$

$$\chi_{\boldsymbol{\phi}}^{\{\bar{\mu};\lambda\}} = \sum_{\zeta\vdash z} (-1)^{z} \sum_{\boldsymbol{p},\boldsymbol{q}} K_{\boldsymbol{q}}^{\mu/\zeta} K_{\boldsymbol{p}}^{\lambda/\tilde{\zeta}}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{\phi}}, \tag{4.30}$$

O($2k+1$)
$$\chi_{\boldsymbol{\phi}}^{[\lambda]} = \sum_{\epsilon\vdash e} (-1)^{(e-\varepsilon)/2} \sum_{\boldsymbol{p},\boldsymbol{q}} K_{\boldsymbol{p};\boldsymbol{q}}^{\lambda/\epsilon}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{\phi}}, \tag{4.31}$$

$$\chi_{\boldsymbol{\phi}}^{[\Delta;\lambda]} = \sum_{\alpha\vdash a} (-1)^{a/2} \sum_{\boldsymbol{p},\boldsymbol{q}} K_{\boldsymbol{p};\boldsymbol{q}}^{\lambda/\alpha} \sum_{\boldsymbol{d}} \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{q}+\boldsymbol{d})\cdot\boldsymbol{\phi}}, \tag{4.32}$$

Sp($2k$)
$$\chi_{\boldsymbol{\phi}}^{\langle\lambda\rangle} = \sum_{\alpha\vdash a} (-1)^{a/2} \sum_{\boldsymbol{p},\boldsymbol{q}} K_{\boldsymbol{p};\boldsymbol{q}}^{\lambda/\alpha}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{\phi}}, \tag{4.33}$$

O($2k$)
$$\chi_{\boldsymbol{\phi}}^{[\lambda]} = \sum_{\gamma\vdash c} (-1)^{c/2} \sum_{\boldsymbol{p},\boldsymbol{q}} K_{\boldsymbol{p};\boldsymbol{q}}^{\lambda/\gamma}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{\phi}}, \tag{4.34}$$

$$\chi_{\boldsymbol{\phi}}^{[\Delta;\lambda]} = \sum_{\epsilon\vdash e} (-1)^{(e+r)/2} \sum_{\boldsymbol{p},\boldsymbol{q}} K_{\boldsymbol{k};\boldsymbol{q}}^{\lambda/\epsilon} \sum_{\boldsymbol{d}} \mathrm{e}^{\mathrm{i}(\boldsymbol{p}-\boldsymbol{q}+\boldsymbol{d})\cdot\boldsymbol{\phi}}. \tag{4.35}$$

where $\boldsymbol{d}$ is summed over all vectors with components either $+\frac{1}{2}$ or $-\frac{1}{2}$ whilst $\boldsymbol{p}$ and $\boldsymbol{q}$ are summed over all vectors with non-negative integer components, and the notation used is such that $\boldsymbol{p};\boldsymbol{q}$ denotes the vector $(p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k)$ defined through the adjunction of the components of the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$.

5. The explicit evaluation of weight multiplicities

The exploitation of the character formulae (4.29)–(4.35) to determine all weight vectors and to evaluate the multiplicity of each weight is a straightforward task which is considerably eased by the use of the symmetry properties (3.2) and (3.7). It is easy to see that these lead immediately to the symmetries of the weight diagrams mentioned in § 2 through the invariance of the appropriate multiplicities under arbitrary permutations of the components of the vectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{d}$ and $\boldsymbol{p};\boldsymbol{q}$, and under an arbitrary number of independent sign changes of the components of the vector $\boldsymbol{d}$.

It is therefore only necessary to evaluate the multiplicities of the leading weights denoted by:

U(k)	$M_{(\sigma)}^{\{\lambda\}}$,	$M_{(\bar{\tau};\sigma)}^{\{\bar{\mu};\lambda\}}$,	(5.1)
O($2k+1$)	$M_{(\sigma)}^{[\lambda]}$,	$M_{(\Delta;\sigma)}^{[\Delta;\lambda]}$,	(5.2)
Sp($2k$)	$M_{(\sigma)}^{\langle\lambda\rangle}$,		(5.3)
O($2k$)	$M_{(\sigma)}^{[\lambda]}$,	$M_{(\Delta;\sigma)}^{[\Delta;\lambda]}$	(5.4)

where $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ are the partitions $(\sigma_1, \sigma_2, \ldots, \sigma_u)$ and $(\tau_1, \tau_2, \ldots \tau_v)$ whilst $(\boldsymbol{\sigma})$, $(\bar{\boldsymbol{\tau}}; \boldsymbol{\sigma})$ and $(\boldsymbol{\Delta}; \boldsymbol{\sigma})$ denote the vectors $(\sigma_1, \sigma_2, \ldots, \sigma_u, 0, 0, \ldots, 0)$, $(\sigma_1, \sigma_2, \ldots, \sigma_u, 0, 0, \ldots, 0, -\tau_v, \ldots -\tau_2, -\tau_1)$ and $(\sigma_1 + \frac{1}{2}, \sigma_2 + \frac{1}{2}, \ldots, \sigma_u + \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ respectively.

The number of distinct weights having these multiplicities (5.1)–(5.4) may be found by considering permutations and sign changes of the components. If $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ may be written as $(\sigma_1^{s_1} \sigma_{s_1+1}^{s_2} \ldots)$ and $(\tau_1^{t_1}, \tau_{t_1+1}^{t_2}, \ldots)$ with $s_1 + s_2 + \ldots = u$ and $t_1 + t_2 + \ldots = v$ then the number of weights connected by these Weyl symmetry operations are given by:

$$\text{U}(k) \qquad (\boldsymbol{\sigma}):\ \binom{k}{s_1 s_2 \ldots} \qquad (\bar{\boldsymbol{\tau}}; \boldsymbol{\sigma}):\ \binom{k}{s_1 s_2 \ldots t_1 t_2 \ldots} \tag{5.5}$$

$$\text{O}(2k+1) \qquad (\boldsymbol{\sigma}):\ 2^u \binom{k}{s_1 s_2 \ldots} \qquad (\boldsymbol{\Delta}; \boldsymbol{\sigma}):\ 2^k \binom{k}{s_1 s_2 \ldots} \tag{5.6}$$

$$\text{Sp}(2k) \qquad (\boldsymbol{\sigma}):\ 2^u \binom{k}{s_1 s_2 \ldots} \tag{5.7}$$

$$\text{O}(2k) \qquad (\boldsymbol{\sigma}):\ 2^u \binom{k}{s_1 s_2 \ldots} \qquad (\boldsymbol{\Delta}; \boldsymbol{\sigma}):\ 2^k \binom{k}{s_1 s_2 \ldots} \tag{5.8}$$

where the multimonial coefficients are defined by:

$$\binom{k}{s_1 s_2 \ldots} = \frac{k!}{(k-u)!} \frac{1}{s_1! s_2! \ldots} \qquad \text{with } s_1 + s_2 + \ldots = u. \tag{5.9}$$

It is to be noted that

$$\text{U}(k) \qquad M_{(\bar{\boldsymbol{\tau}}; \boldsymbol{\sigma})}^{\{\bar{\boldsymbol{\mu}}; \boldsymbol{\lambda}\}} = M_{(\bar{\boldsymbol{\sigma}}; \boldsymbol{\tau})}^{\{\bar{\boldsymbol{\lambda}}; \boldsymbol{\mu}\}} \tag{5.10}$$

as may be seen from the formula (4.30).

From (4.29)–(4.35) it follows that the multiplicities of the leading weights (5.1)–(5.4) are given by:

$$\text{U}(k) \qquad M_{(\boldsymbol{\sigma})}^{\{\boldsymbol{\lambda}\}} = K_{\boldsymbol{\sigma}}^{\boldsymbol{\lambda}}, \tag{5.11}$$

$$M_{(\bar{\boldsymbol{\tau}}; \boldsymbol{\sigma})}^{\{\bar{\boldsymbol{\mu}}; \boldsymbol{\lambda}\}} = \sum_{\zeta \vdash z} \sum_{\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}} (-1)^z K_{\boldsymbol{f}; \boldsymbol{\tau}+\boldsymbol{g}; \boldsymbol{h}}^{\boldsymbol{\mu}/\zeta} K_{\boldsymbol{\sigma}+\boldsymbol{f}; \boldsymbol{g}; \boldsymbol{h}}^{\boldsymbol{\lambda}/\zeta}, \tag{5.12}$$

$$\text{O}(2k+1) \qquad M_{(\boldsymbol{\sigma})}^{[\boldsymbol{\lambda}]} = \sum_{\epsilon \vdash e} \sum_{\boldsymbol{f}, \boldsymbol{h}} (-1)^{(e-r)/2} K_{\boldsymbol{\sigma}+\boldsymbol{f}; \boldsymbol{f}; \boldsymbol{h}; \boldsymbol{h}}^{\boldsymbol{\lambda}/\epsilon}, \tag{5.13}$$

$$M_{(\boldsymbol{\Delta}; \boldsymbol{\sigma})}^{[\boldsymbol{\Delta}\, \boldsymbol{\lambda}]} = \sum_{\alpha \vdash a} \sum_{\boldsymbol{f}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{j}} (-1)^{a/2} K_{\boldsymbol{\sigma}+\boldsymbol{f}+\boldsymbol{i}; \boldsymbol{f}; \boldsymbol{h}+\boldsymbol{j}; \boldsymbol{h}}^{\boldsymbol{\lambda}/\alpha}. \tag{5.14}$$

$$\text{Sp}(2k) \qquad M_{(\boldsymbol{\sigma})}^{\langle \boldsymbol{\lambda} \rangle} = \sum_{\alpha \vdash a} \sum_{\boldsymbol{f}, \boldsymbol{h}} (-1)^{a/2} K_{\boldsymbol{\sigma}+\boldsymbol{f}; \boldsymbol{f}; \boldsymbol{h}; \boldsymbol{h}}^{\boldsymbol{\lambda}/\alpha}. \tag{5.15}$$

$$\text{O}(2k) \qquad M_{(\boldsymbol{\sigma})}^{[\boldsymbol{\lambda}]^{\pm}} = \sum_{\gamma \vdash c} \sum_{\boldsymbol{f}, \boldsymbol{h}} (-1)^{c/2} K_{\boldsymbol{\sigma}+\boldsymbol{f}; \boldsymbol{f}; \boldsymbol{h}; \boldsymbol{h}}^{\boldsymbol{\lambda}/\gamma}, \tag{5.16}$$

$$M_{(\boldsymbol{\Delta}; \boldsymbol{\sigma})}^{[\boldsymbol{\Delta}; \boldsymbol{\lambda}]} = \sum_{\epsilon \vdash e} \sum_{\boldsymbol{f}, \boldsymbol{h}, \boldsymbol{i}, \boldsymbol{j}} (-1)^{(e+r)/2} K_{\boldsymbol{\sigma}+\boldsymbol{f}+\boldsymbol{i}; \boldsymbol{f}; \boldsymbol{h}+\boldsymbol{j}; \boldsymbol{h}}^{\boldsymbol{\lambda}/\epsilon}, \tag{5.17}$$

where now $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$ are summed over all u-, v-, and w-dimensional vectors with non-negative integer components, whilst $\boldsymbol{i}$ and $\boldsymbol{j}$ are summed over all u- and w-dimensional vectors with all components either 0 or 1. In (5.12) $w = k - u - v$, whilst in (5.13)–(5.17) $w = k - u$. The notation adopted is such that in (5.17), for example, the vector $(\boldsymbol{\sigma} + \boldsymbol{f} + \boldsymbol{i}; \boldsymbol{f}; \boldsymbol{h} + \boldsymbol{j}; \boldsymbol{h})$ has components $(\sigma_1 + f_1 + i_1, \sigma_2 + f_2 + i_2, \ldots \sigma_u + f_u +$

$i_u, f_1, f_2, \ldots, f_u, h_1+j_1, h_2+j_2, \ldots, h_w+j_w, h_1, h_2, \ldots h_w)$ with $w=k-u$. Thus the symbols + and ; denote vector addition and the adjunction of components respectively.

In these expressions (5.11)–(5.17) it is to be noted that the k dependence of the weight multiplicities comes about through the summations over the vectors $\boldsymbol{h}$ and $\boldsymbol{j}$. Indeed in (5.12) the summation over $\boldsymbol{h}$ may be replaced by a summation over arbitrary partitions $\boldsymbol{\omega}$ provided that for each partition of the form $(\omega_1^{w_1}, \omega_{w_1+1}^{w_2} \ldots)$ the result is multiplied by the multinomial coefficient

$$\binom{k-u-v}{w_1 w_2 \ldots} \tag{5.18}$$

representing the number of vectors $\boldsymbol{h}$ related to $\boldsymbol{\omega}$ by permutations of its components.

In the same way in (5.13) and (5.16) the summation over $\boldsymbol{h}$ may be replaced by a summation over the special partitions $\boldsymbol{\beta}$ provided that for each such partition of the form $(\beta_1^{2b_1}, \beta_{2b_1+1}^{2b_2}, \ldots)$ the result is multiplied by the multinomial coefficient

$$\binom{k-u}{b_1 b_2 \ldots}. \tag{5.19}$$

The summations over $\boldsymbol{h}$ and $\boldsymbol{j}$ in (5.14) and (5.16) are more complicated to deal with, but the $\boldsymbol{h}$ summation may be replaced by a summation over $\boldsymbol{\beta}$ with the inclusion of the additional factor (5.19), whilst the summation over $\boldsymbol{j}$ is accomplished by adding 0 or 1 in all possible ways to alternate parts of $\boldsymbol{\beta}$ starting with the first part. This will lead to a further dependence on k if the vectors so obtained are replaced by partitions.

The simplest formula of all is of course (5.11). By virtue of the definition of Kostka's matrix, (3.4) gives rise to multiplicities $M_{(\sigma)}^{\{\lambda\}}$ of U(k) which are independent of k. The use of this formula corresponds to the use of a procedure due to Delaney and Gruber (1969) and has been discussed elsewhere (King and Plunkett 1972). It is to be noted that the use of (3.12) is particularly advantageous since it gives elements of the multiplicity matrix column by column. For example, in the notation appropriate to representations of U(k):

$$\{2\}\times\{1\}\times\{1\}=\{4\}+2\{31\}+\{2^2\}+\{21^2\} \tag{5.20}$$

corresponding to the existence of the standard Young tableaux:

1123	112	113	11	11.
	3	2	23	2
				3

Hence

$$\mathrm{U}(k) \qquad M_{(21^2)}^{\{4\}}=1 \quad M_{(21^2)}^{\{31\}}=2 \quad M_{(21^2)}^{\{2^2\}}=1 \quad M_{(21^2)}^{\{21^2\}}=1 \quad M_{(21^2)}^{\{1^4\}}=0. \tag{5.21}$$

In order to use the formulae (5.12), (5.13), (5.15) and (5.16) it is first necessary to evaluate some S-function quotients. For example

$$\sum_{\zeta\vdash z}(-1)^z\{\overline{1^3/\zeta}\}\times\{2/\tilde{\zeta}\}=\{\overline{1^3}\}\times\{2\}-\{\overline{1^2}\}\times\{1\}+\{\overline{1}\}\times\{0\}, \tag{5.22}$$

$$\sum_{\boldsymbol{\epsilon}\vdash e}(-1)^{(e-r)/2}\{31/\boldsymbol{\epsilon}\}=\{31\}+\{3\}+\{21\}-\{1\}, \tag{5.23}$$

$$\sum_{\boldsymbol{\alpha}\vdash a}(-1)^{a/2}\{31/\boldsymbol{\alpha}\}=\{31\}-\{2\}, \tag{5.24}$$

$$\sum_{\gamma\vdash c}(-1)^{c/2}\{31/\gamma\}=\{31\}-\{2\}-\{1^2\}+\{0\}. \tag{5.25}$$

The k dependence (5.18) arising from the use of (5.12) may then be illustrated by, for example, the calculation of the multiplicities $M^{\{\bar{1}^3;2\}}_{(\bar{1}^2;1)}$ of U(k). The first term of (5.22) gives the contribution

$$\sum_{f,g,\omega} K^{\{1^3\}}_{f;1^2+g,\omega}K^{\{2\}}_{1+f;g,\omega}\binom{k-3}{w_1w_2\ldots}$$

$$=K^{\{1^3\}}_{0;1^2;1}K^{\{2\}}_{1;0;1}\binom{k-3}{1}+K^{\{1^3\}}_{1;1^2;0}K^{\{2\}}_{2;0;0}+K^{\{1^3\}}_{0;21;0}K^{\{2\}}_{1;10;0}+K^{\{1^3\}}_{0;12;0}K^{\{2\}}_{1;01;0}$$

$$=1\,.\,1\,.\,(k-3)+1\,.\,1\,.+0\,.\,1+0\,.\,1=k-2. \tag{5.26}$$

The contribution of the second term of (5.22) is easily seen to be -1, whilst the third term gives no contribution. Hence

$$\text{U}(k) \qquad M^{\{\bar{1}^3;2\}}_{(\bar{1}^2;1)}=k-3. \tag{5.27}$$

In the same way the contribution of the leading term {31} in (5.23), (5.24) and (5.25) to the multiplicities $M^{[31]}_{(2)}$, $M^{\langle 31\rangle}_{(2)}$, and $M^{[31]}_{(2)}$ of the groups O($2k+1$), Sp($2k$) and O($2k$) respectively, are all given by:

$$\sum_{f,\beta} K^{31}_{2+f;f;\beta}\binom{k-1}{b_1b_2\ldots}=K^{31}_{2;0;1^2}\binom{k-1}{1}+K^{31}_{3;1;0}=2(k-1)+1=2k-1. \tag{5.28}$$

The only other terms giving rise to non-vanishing contributions to these multiplicities are the terms $-\{2\}$ in (5.24) and (5.25). In each case the contribution is easily seen to be -1. Combining these results yields the multiplicities:

$$\text{O}(2k+1) \qquad M^{[31]}_{(2)}=2k-1, \tag{5.29}$$

$$\text{Sp}(2k) \qquad M^{\langle 31\rangle}_{(2)}=2k-2, \tag{5.30}$$

$$\text{O}(2k) \qquad M^{[31]}_{(2)}=2k-2. \tag{5.31}$$

As a final example the multiplicities $M^{[\Delta;31]}_{(\Delta;2)}$ of the groups O($2k+1$) and O($2k$) may be found from (5.14) and (5.17) respectively by first making use of the S-function quotients:

$$\sum_{\alpha\vdash a}(-1)^{a/2}\{31/\alpha\}=\{31\}-\{2\} \tag{5.32}$$

$$\sum_{\epsilon\vdash e}(-1)^{(e+r)/2}\{31/\epsilon\}=\{31\}-\{3\}-\{21\}+\{1\}. \tag{5.33}$$

The contributions of the leading term {31} to the multiplicities are:

$$\sum_{f,h,i,j} K^{31}_{2+f+i;f;h+j;h}$$

$$=K^{31}_{2;0;1;1}\binom{k-1}{1}+K^{31}_{2;0;1^2;0}\binom{k-1}{2}+K^{31}_{3;0;1;0}\binom{k-1}{1}+K^{31}_{3;1;0;0}$$

$$=2(k-1)+2\,.\,\tfrac{1}{2}(k-1)(k-2)+(k-1)+1=k^2. \tag{5.34}$$

Similarly the contributions of the terms {3} and {21} are:

$$\sum_{f,h,i,j} K^{3}_{2+f+i;f;h+j;h} = K^{3}_{2;0;1;0}\binom{k-1}{1} + K^{3}_{3;0;0;0} = (k-1)+1 = k, \tag{5.35}$$

and

$$\sum_{f,h,i,j} K^{21}_{2+f+i;f;h+j;h} = K^{21}_{2;0;1;0}\binom{k-1}{1} = k-1, \tag{5.36}$$

whilst that of {2} is clearly 1. Combining these results yields the multiplicities:

$$\mathrm{O}(2k+1) \qquad M^{[\Delta;31]}_{(\Delta;2)} = k^2-1, \tag{5.37}$$

$$\mathrm{O}(2k) \qquad M^{[\Delta;31]}_{(\Delta;2)} = k^2-2k+1. \tag{5.38}$$

6. Tabulation of results

It has been shown that Littlewood's use of S-function techniques leading to explicit formulae for characters of representations of the classical Lie groups may be further exploited to evaluate weight multiplicities. The evaluation procedure depends upon the determination of the elements K^{λ}_{μ} of the Kostka matrix, together with some combinatorics of vectors with integer-valued components. The only other necessary ingredient is the determination of certain S-function quotients associated with some well known series of S-functions.

The results obtained in this way are given in tables 1–7. They may be checked for consistency in several ways. Use may be made of the factors (5.5)–(5.8) to check the dimensions of the representations against known formulae (Robinson 1961, p 60, King 1970, Abramsky *et al* 1973). For example the dimension of the representation $\{\bar{1}^3; 21\}$ of U(k) is given by:

$$D_k\{\bar{1}^3; 2\} = 1\binom{k}{3\;\;1} + 1\binom{k}{3\;\;2} + (k-3)\binom{k}{2\;\;1} + \tfrac{1}{2}(k^2-5k+6)\binom{k}{1}, \tag{6.1}$$

where the non-vanishing multiplicities:

$$M^{\{\bar{1}^3;2\}}_{(\bar{1}^3;2)} = 1 \qquad M^{\{\bar{1}^3;2\}}_{(\bar{1}^3;1^2)} = 1 \qquad M^{\{\bar{1}^3;2\}}_{(\bar{1}^2;1)} = k-3 \qquad M^{\{\bar{1}^3;2\}}_{(\bar{1};0)} = \tfrac{1}{2}(k^2-5k+6), \tag{6.2}$$

have been taken from table 2 ıfter making use of (5.10). It then follows that

$$D_k\{\bar{1}^3; 2\} = \tfrac{1}{12}(k-3)(k-2)k(k+1)(k+2) \tag{6.3}$$

in accordance with the general formula given elsewhere (King 1970).

Alternatively, other checks are provided by the use of the restriction formulae (4.7), (4.8) and (King 1975):

$$\mathrm{O}(2k) \downarrow \mathrm{U}(k) \qquad [\lambda] \downarrow \sum_{\zeta,\beta} \{\bar{\zeta}; \lambda/\zeta\beta\} \tag{6.4}$$

$$[\Delta; \lambda] \downarrow \epsilon^{-\frac{1}{2}} \sum_{\zeta,\beta,m} \{\bar{\zeta}; (\lambda/\zeta\beta)\cdot\tilde{m}\} \tag{6.5}$$

$$\mathrm{Sp}(2k) \downarrow \mathrm{U}(k) \qquad \langle\lambda\rangle \downarrow \sum_{\zeta,\delta} \{\bar{\zeta}; \lambda/\zeta\delta\} \tag{6.6}$$

Table 1. Weight multiplicities $M^{\{\lambda\}}_{(\sigma)}$ of U(k).

{λ} \ (σ)	(0)
{0}	1

	(1)
{1}	1

	(2)	(1^2)
{2}	1	1
{1^2}		1

	(3)	(21)	(1^3)
{3}	1	1	1
{21}		1	2
{1^3}			1

	(4)	(31)	(2^2)	(21^2)	(1^4)
{4}	1	1	1	1	1
{31}		1	1	2	3
{2^2}			1	1	2
{21^2}				1	3
{1^4}					1

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
{5}	1	1	1	1	1	1	1
{41}		1	1	2	2	3	4
{32}			1	1	2	3	5
{31^2}				1	1	3	6
{2^21}					1	2	5
{21^3}						1	4
{1^5}							1

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)
{6}	1	1	1	1	1	1	1	1	1	1	1
{51}		1	1	2	1	2	3	2	3	4	5
{42}			1	1	1	2	3	3	4	6	9
{41^2}				1	0	1	3	1	3	6	10
{3^2}					1	1	1	1	2	3	5
{321}						1	2	2	4	8	16
{31^3}							1	0	1	4	10
{2^3}								1	1	2	5
{2^21^2}									1	3	9
{21^4}										1	5
{1^6}											1

Table 2. Weight multiplicities $M^{\{\bar{\mu};\lambda\}}_{(\bar{\tau};\sigma)}$ of U(k).

$\{\bar{\mu};\lambda\}$ \ $(\bar{\tau};\sigma)$	$(\bar{1};1)$	(0)
$\{\bar{1};1\}$	1	$k-1$

	$(\bar{1};2)$	$(\bar{1};1^2)$	(1)
$\{\bar{1};2\}$	1	1	$k-1$
$\{\bar{1};1^2\}$		1	$k-2$

	$(\bar{1};3)$	$(\bar{1};21)$	$(\bar{1};1^3)$	(2)	(1^2)
$\{\bar{1};3\}$	1	1	1	$k-1$	$k-1$
$\{\bar{1};21\}$		1	2	$k-2$	$2k-4$
$\{\bar{1};1^3\}$			1		$k-3$

	$(\bar{2};2)$	$(\bar{2};1^2)$	$(\bar{1}^2;2)$	$(\bar{1}^2;1^2)$	$(\bar{1};1)$	(0)
$\{\bar{2};2\}$	1	1	1	1	$k-1$	$\frac{1}{2}(k^2-k)$
$\{\bar{2};1^2\}$		1		1	$k-2$	$\frac{1}{2}(k^2-3k+2)$
$\{\bar{1}^2;2\}$			1	1	$k-2$	$\frac{1}{2}(k^2-3k+2)$
$\{\bar{1}^2;1^2\}$				1	$k-3$	$\frac{1}{2}(k^2-3k)$

	$(\bar{1};4)$	$(\bar{1};31)$	$(\bar{1};2^2)$	$(\bar{1};21^2)$	$(\bar{1};1^4)$	(3)	(21)	(1^3)
$\{\bar{1};4\}$	1	1	1	1	1	$k-1$	$k-1$	$k-1$
$\{\bar{1};31\}$		1	1	2	3	$k-2$	$2k-4$	$3k-6$
$\{\bar{1};2^2\}$			1	1	2		$k-2$	$2k-5$
$\{\bar{1};21^2\}$				1	3		$k-3$	$3k-9$
$\{\bar{1};1^4\}$					1			$k-4$

	$(\bar{2};3)$	$(\bar{2};21)$	$(\bar{2};1^3)$	$(\bar{1}^2;3)$	$(\bar{1}^2;21)$	$(\bar{1}^2;1^3)$	$(\bar{1};2)$	$(\bar{1};1^2)$	(1)
$\{\bar{2};3\}$	1	1	1	1	1	1	$k-1$	$k-1$	$\frac{1}{2}(k^2-k)$
$\{\bar{2};21\}$		1	2		1	2	$k-2$	$2k-4$	$\frac{1}{2}(2k^2-6k+4)$
$\{\bar{2};1^3\}$			1			1		$k-3$	$\frac{1}{2}(k^2-5k+6)$
$\{\bar{1}^2;3\}$				1	1	1	$k-2$	$k-2$	$\frac{1}{2}(k^2-3k+2)$
$\{\bar{1}^2;21\}$					1	2	$k-3$	$2k-6$	$\frac{1}{2}(2k^2-8k+6)$
$\{\bar{1}^2;1^3\}$						1		$k-4$	$\frac{1}{2}(k^2-5k+4)$

Table 2—continued.

	$(\bar{1};5)$	$(\bar{1};41)$	$(\bar{1};32)$	$(\bar{1};31^2)$	$(\bar{1};2^21)$	$(\bar{1};21^3)$	$(\bar{1};1^5)$	(4)	(31)	(2^2)	(21^2)	(1^4)
$\{\bar{1};5\}$	1	1	1	1	1	1	1	$k-1$	$k-1$	$k-1$	$k-1$	$k-1$
$\{\bar{1};41\}$		1	1	2	2	3	4	$k-2$	$2k-4$	$2k-4$	$3k-6$	$4k-8$
$\{\bar{1};32\}$			1	1	2	3	5		$k-2$	$2k-4$	$3k-7$	$5k-13$
$\{\bar{1};31^2\}$				1	1	3	6		$k-3$	$k-3$	$3k-9$	$6k-18$
$\{\bar{1};2^21\}$					1	2	5			$k-3$	$2k-6$	$5k-17$
$\{\bar{1};21^3\}$						1	4				$k-4$	$4k-16$
$\{\bar{1};1^5\}$							1					$k-5$

	$(\bar{2};4)$	$(\bar{2};31)$	$(\bar{2};2^2)$	$(\bar{2};21^2)$	$(\bar{2};1^4)$	$(\bar{1}^2;4)$	$(\bar{1}^2;31)$	$(\bar{1}^2;2^2)$	$(\bar{1}^2;21^2)$	$(\bar{1}^2;1^4)$	$(\bar{1};3)$	$(\bar{1};21)$	$(\bar{1};1^3)$	(2)	(1^2)
$\{\bar{2};4\}$	1	1	1	1	1	1	1	1	1	1	$k-1$	$k-1$	$k-1$	$\frac{1}{2}(k^2-k)$	$\frac{1}{2}(k^2-k)$
$\{\bar{2};31\}$		1	1	2	3		1	1	2	3	$k-2$	$2k-4$	$3k-6$	$\frac{1}{2}(2k^2-6k+4)$	$\frac{1}{2}(3k^2-9k+6)$
$\{\bar{2};2^2\}$			1	1	2			1	1	2		$k-2$	$2k-5$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{2}(2k^2-8k+8)$
$\{\bar{2};21^2\}$				1	3				1	3		$k-3$	$3k-9$	$\frac{1}{2}(k^2-5k+6)$	$\frac{1}{2}(3k^2-15k+18)$
$\{\bar{2};1^4\}$					1					1			$k-4$		$\frac{1}{2}(k^2-7k+12)$
$\{\bar{1}^2;4\}$						1	1	1	1	1	$k-2$	$k-2$	$k-2$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{2}(k^2-3k+2)$
$\{\bar{1}^2;31\}$							1	1	2	3	$k-3$	$2k-6$	$3k-9$	$\frac{1}{2}(2k^2-8k+6)$	$\frac{1}{2}(3k^2-13k+12)$
$\{\bar{1}^2;2^2\}$								1	1	2		$k-3$	$2k-7$	$\frac{1}{2}(k^2-5k+6)$	$\frac{1}{2}(2k^2-10k+12)$
$\{\bar{1}^2;21^2\}$									1	3		$k-4$	$3k-12$	$\frac{1}{2}(k^2-5k+4)$	$\frac{1}{2}(3k^2-17k+20)$
$\{\bar{1}^2;1^4\}$										1			$k-5$		$\frac{1}{2}(k^2-7k+10)$

	$(\bar{3};3)$	$(\bar{3};21)$	$(\bar{3};1^3)$	$(\overline{21};3)$	$(\overline{21};21)$	$(\overline{21};1^3)$	$(\bar{1}^3;3)$	$(\bar{1}^3;21)$	$(\bar{1}^3;1^3)$	$(\bar{2};2)$	$(\bar{2};1^2)$	$(\bar{1}^2;2)$	$(\bar{1}^2;1^2)$	$(\bar{1};1)$	(0)
$\{\bar{3};3\}$	1	1	1	1	1	1	1	1	1	$k-1$	$k-1$	$k-1$	$k-1$	$\frac{1}{2}(k^2-k)$	$\frac{1}{6}(k^3-k)$
$\{\bar{3};21\}$		1	2		1	2		1	2	$k-2$	$2k-4$	$k-2$	$2k-4$	$\frac{1}{2}(2k^2-6k+4)$	$\frac{1}{6}(2k^3-6k^2+4k)$
$\{\bar{3};1^3\}$			1			1			1		$k-3$		$k-3$	$\frac{1}{2}(k^2-5k+6)$	$\frac{1}{6}(k^3-6k^2+11k-6)$
$\{\overline{21};3\}$				1	1	1	2	2	2	$k-2$	$k-2$	$2k-4$	$2k-4$	$\frac{1}{2}(2k^2-6k+4)$	$\frac{1}{6}(2k^3-6k^2+4k)$
$\{\overline{21};21\}$					1	2		2	4	$k-3$	$2k-6$	$2k-6$	$4k-12$	$\frac{1}{2}(4k^2-18k+18)$	$\frac{1}{6}(4k^3-18k^2+20k-6)$
$\{\overline{21};1^3\}$						1			2		$k-4$		$2k-8$	$\frac{1}{2}(2k^2-12k+16)$	$\frac{1}{6}(2k^3-12k^2+16k)$
$\{\bar{1}^3;3\}$							1	1	1			$k-3$	$k-3$	$\frac{1}{2}(k^2-5k+6)$	$\frac{1}{6}(k^3-6k^2+11k-6)$
$\{\bar{1}^3;21\}$								1	2			$k-4$	$2k-8$	$\frac{1}{2}(2k^2-12k+16)$	$\frac{1}{6}(2k^3-12k^2+16k)$
$\{\bar{1}^3;1^3\}$									1				$k-5$	$\frac{1}{2}(k^2-7k+10)$	$\frac{1}{6}(k^3-6k^2+5k)$

Table 3. Weight multiplicities $M^{[\lambda]}_{(\sigma)}$ of $O(2k+1)$.

$[\lambda]$ \ (σ)	(0)
[0]	1

	(1)	(0)
[1]	1	1

	(2)	(1^2)	(1)	(0)
[2]	1	1	1	k
$[1^2]$		1	1	k

	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
[3]	1	1	1	1	1	k	k
[21]		1	2	1	2	$2k-1$	$2k-1$
$[1^3]$			1		1	$k-1$	k

	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
[4]	1	1	1	1	1	1	1	1	k	k	k	$\frac{1}{2}(k^2+k)$
[31]		1	1	2	3	1	2	3	$2k-1$	$3k-2$	$3k-2$	$\frac{1}{2}(3k^2-k)$
$[2^2]$			1	1	2		1	2	$k-1$	$2k-2$	$2k-2$	$\frac{1}{2}(2k^2-2)$
$[21^2]$				1	3		1	3	$k-1$	$3k-4$	$3k-3$	$\frac{1}{2}(3k^2-3k)$
$[1^4]$					1			1		$k-2$	$k-1$	$\frac{1}{2}(k^2-k)$

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
[5]	1	1	1	1	1	1	1	1	1	1	1	1	k	k	k	k	k	$\frac{1}{2}(k^2+k)$	$\frac{1}{2}(k^2+k)$
[41]		1	1	2	2	3	4	1	2	2	3	4	$2k-1$	$3k-2$	$4k-3$	$3k-2$	$4k-3$	$\frac{1}{2}(4k^2-2k)$	$\frac{1}{2}(4k^2-2k)$
[32]			1	1	2	3	5		1	2	3	5	$k-1$	$3k-3$	$5k-6$	$3k-3$	$5k-6$	$\frac{1}{2}(5k^2-5k)$	$\frac{1}{2}(5k^2-5k)$
$[31^2]$				1	1	3	6		1	1	3	6	$k-1$	$3k-4$	$6k-9$	$3k-3$	$6k-8$	$\frac{1}{2}(6k^2-10k+4)$	$\frac{1}{2}(6k^2-8k+2)$
$[2^21]$					1	2	5			1	2	5		$2k-3$	$5k-9$	$2k-3$	$5k-8$	$\frac{1}{2}(5k^2-9k+2)$	$\frac{1}{2}(5k^2-7k)$
$[21^3]$						1	4				1	4		$k-2$	$4k-9$	$k-1$	$4k-7$	$\frac{1}{2}(4k^2-10k+6)$	$\frac{1}{2}(4k^2-6k)$
$[1^5]$							1					1			$k-3$		$k-2$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{2}(k^2-k)$

Table 3—continued.

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)
[6]	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
[51]		1	1	2	1	2	3	2	3	4	5	1	2	2	3	3	4	5
[42]			1	1	1	2	3	3	4	6	9		1	2	3	4	6	9
[41^2]				1	0	1	3	1	3	6	10		1	1	3	3	6	10
[3^2]					1	1	1	1	2	3	5			1	1	2	3	5
[321]						1	2	2	4	8	16			1	2	4	8	16
[31^3]							1	0	1	4	10				1	1	4	10
[2^3]								1	1	2	5					1	2	5
[2^21^2]									1	3	9					1	3	9
[21^4]										1	5						1	5
[1^6]											1							1

	(4)	(31)	(2^2)	(21^2)	(1^4)	(3)	(21)	(1^3)	(2)	(1^2)	(1)	(0)
[6]	k	k	k	k	k	k	k	k	$\frac{1}{2}(k^2+k)$	$\frac{1}{2}(k^2+k)$	$\frac{1}{2}(k^2+k)$	$\frac{1}{6}(k^3+3k^2+2k)$
[51]	$2k-1$	$3k-2$	$3k-2$	$4k-3$	$5k-4$	$3k-2$	$4k-3$	$5k-4$	$\frac{1}{2}(4k^2-2k)$	$\frac{1}{2}(5k^2-3k)$	$\frac{1}{2}(5k^2-3k)$	$\frac{1}{6}(5k^3+3k^2-2k)$
[42]	$k-1$	$3k-3$	$4k-4$	$6k-7$	$9k-12$	$3k-3$	$6k-7$	$9k-12$	$\frac{1}{2}(6k^2-6k)$	$\frac{1}{2}(9k^2-13k+4)$	$\frac{1}{2}(9k^2-13k+4)$	$\frac{1}{6}(9k^3-3k^2-6k)$
[41^2]	$k-1$	$3k-4$	$3k-4$	$6k-9$	$10k-16$	$3k-3$	$6k-8$	$10k-15$	$\frac{1}{2}(6k^2-10k+4)$	$\frac{1}{2}(10k^2-20k+10)$	$\frac{1}{2}(10k^2-18k+8)$	$\frac{1}{6}(10k^3-12k^2+2k)$
[3^2]		$k-1$	$2k-2$	$3k-4$	$5k-8$	$k-1$	$3k-4$	$5k-8$	$\frac{1}{2}(3k^2-5k+2)$	$\frac{1}{2}(5k^2-9k+4)$	$\frac{1}{2}(5k^2-9k+4)$	$\frac{1}{6}(5k^3-3k^2-2k)$
[321]		$2k-3$	$4k-6$	$8k-14$	$16k-32$	$2k-3$	$8k-13$	$16k-30$	$\frac{1}{2}(8k^2-16k+6)$	$\frac{1}{2}(16k^2-40k+24)$	$\frac{1}{2}(16k^2-36k+18)$	$\frac{1}{6}(16k^3-24k^2-4k+6)$
[31^3]		$k-2$	$k-2$	$4k-9$	$10k-24$	$k-1$	$4k-7$	$10k-21$	$\frac{1}{2}(4k^2-10k+6)$	$\frac{1}{2}(10k^2-32k+26)$	$\frac{1}{2}(10k^2-26k+16)$	$\frac{1}{6}(10k^3-24k^2+14k)$
[2^3]			$k-2$	$2k-4$	$5k-12$		$2k-4$	$5k-11$	$\frac{1}{2}(2k^2-4k)$	$\frac{1}{2}(5k^2-15k+10)$	$\frac{1}{2}(5k^2-13k+6)$	$\frac{1}{6}(5k^3-9k^2-2k)$
[2^21^2]			$k-2$	$3k-7$	$9k-24$		$3k-6$	$9k-21$	$\frac{1}{2}(3k^2-9k+6)$	$\frac{1}{2}(9k^2-31k+26)$	$\frac{1}{2}(9k^2-25k+14)$	$\frac{1}{6}(9k^3-21k^2+6k)$
[21^4]				$k-3$	$5k-16$		$k-2$	$5k-13$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{2}(5k^2-21k+22)$	$\frac{1}{2}(5k^2-15k+10)$	$\frac{1}{6}(5k^3-15k^2+10k)$
[1^6]					$k-4$			$k-3$		$\frac{1}{2}(k^2-5k+6)$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{6}(k^3-3k^2+2k)$

Table 4. Weight multiplicities $M^{[\Delta;\lambda]}_{(\Delta;\sigma)}$ of $O(2k+1)$.

$[\Delta;\lambda]$ \ $(\Delta;\sigma)$	$(\Delta;0)$
$[\Delta;0]$	1

	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;1]$	1	k

	$(\Delta;2)$	$(\Delta;1^2)$	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;2]$	1	1	k	$\frac{1}{2}(k^2+k)$
$[\Delta;1^2]$		1	$k-1$	$\frac{1}{2}(k^2+k-2)$

	$(\Delta;3)$	$(\Delta;21)$	$(\Delta;1^3)$	$(\Delta;2)$	$(\Delta;1^2)$	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;3]$	1	1	1	k	k	$\frac{1}{2}(k^2+k)$	$\frac{1}{6}(k^3+3k^2+2k)$
$[\Delta;21]$		1	2	$k-1$	$2k-2$	$\frac{1}{2}(2k^2-2)$	$\frac{1}{6}(2k^3+6k^2-8k)$
$[\Delta;1^3]$			1		$k-2$	$\frac{1}{2}(k^2-k-2)$	$\frac{1}{6}(k^3+3k^2-10k)$

	$(\Delta;4)$	$(\Delta;31)$	$(\Delta;2^2)$	$(\Delta;21^2)$	$(\Delta;1^4)$	$(\Delta;3)$	$(\Delta;21)$	$(\Delta;1^3)$	$(\Delta;2)$	$(\Delta;1^2)$	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;4]$	1	1	1	1	1	k	k	k	$\frac{1}{2}(k^2+k)$	$\frac{1}{2}(k^2+k)$	$\frac{1}{6}(k^3+3k^2+2k)$	$\frac{1}{24}(k^4+6k^3+11k^2+6k)$
$[\Delta;31]$		1	1	2	3	$k-1$	$2k-2$	$3k-3$	$\frac{1}{2}(2k^2-2)$	$\frac{1}{2}(3k^2-k-2)$	$\frac{1}{6}(3k^3+6k^2-9k)$	$\frac{1}{24}(3k^4+18k^3-3k^2-18k)$
$[\Delta;2^2]$			1	1	2		$k-1$	$2k-3$	$\frac{1}{2}(k^2-k)$	$\frac{1}{2}(2k^2-2k)$	$\frac{1}{6}(2k^3+3k^2-11k+6)$	$\frac{1}{24}(2k^4+12k^3-14k^2)$
$[\Delta;21^2]$				1	3		$k-2$	$3k-6$	$\frac{1}{2}(k^2-k-2)$	$\frac{1}{2}(3k^2-5k-2)$	$\frac{1}{6}(3k^3+3k^2-24k+12)$	$\frac{1}{24}(3k^4+18k^3-39k^2-30k+24)$
$[\Delta;1^4]$					1			$k-3$		$\frac{1}{2}(k^2-3k)$	$\frac{1}{6}(k^3-13k+12)$	$\frac{1}{24}(k^4+6k^3-25k^2-6k)$

Table 5. Weight multiplicities $M^{\langle\lambda\rangle}_{(\sigma)}$ of Sp(2k).

$\langle\lambda\rangle$ \ (σ)	(0)
⟨0⟩	1

	(1)
⟨1⟩	1

	(2)	(1^2)	(0)
⟨2⟩	1	1	k
⟨1^2⟩		1	$k-1$

	(3)	(21)	(1^3)	(1)
⟨3⟩	1	1	1	k
⟨21⟩		1	2	$2k-2$
⟨1^3⟩			1	$k-2$

	(4)	(31)	(2^2)	(21^2)	(1^4)	(2)	(1^2)	(0)
⟨4⟩	1	1	1	1	1	k	k	$\frac{1}{2}(k^2+k)$
⟨31⟩		1	1	2	3	$2k-2$	$3k-3$	$\frac{1}{2}(3k^2-3k)$
⟨2^2⟩			1	1	2	$k-1$	$2k-3$	$\frac{1}{2}(2k^2-2k)$
⟨21^2⟩				1	3	$k-2$	$3k-6$	$\frac{1}{2}(3k^2-7k+2)$
⟨1^4⟩					1		$k-3$	$\frac{1}{2}(k^2-3k)$

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)	(3)	(21)	(1^3)	(0)
⟨5⟩	1	1	1	1	1	1	1	k	k	k	$\frac{1}{2}(k^2+k)$
⟨41⟩		1	1	2	2	3	4	$2k-2$	$3k-3$	$4k-4$	$\frac{1}{2}(4k^2-4k)$
⟨32⟩			1	1	2	3	5	$k-1$	$3k-4$	$5k-8$	$\frac{1}{2}(5k^2-9k+4)$
⟨31^2⟩				1	1	3	6	$k-2$	$3k-6$	$6k-12$	$\frac{1}{2}(6k^2-16k+8)$
⟨2^21⟩					1	2	5		$2k-4$	$5k-12$	$\frac{1}{2}(5k^2-15k+10)$
⟨21^3⟩						1	4		$k-3$	$4k-12$	$\frac{1}{2}(4k^2-16k+12)$
⟨1^5⟩							1			$k-4$	$\frac{1}{2}(k^2-5k+4)$

Table 5—continued.

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)	(4)	(31)	(2^2)	(21^2)	(1^4)
⟨6⟩	1	1	1	1	1	1	1	1	1	1	1	k	k	k	k	k
⟨51⟩		1	1	2	1	2	3	2	3	4	5	$2k-2$	$3k-3$	$3k-3$	$4k-4$	$5k-5$
⟨42⟩			1	1	1	2	3	3	4	6	9	$k-1$	$3k-4$	$4k-5$	$6k-9$	$9k-15$
$\langle 41^2\rangle$				1	0	1	3	1	3	6	10	$k-2$	$3k-6$	$3k-6$	$6k-12$	$10k-20$
$\langle 3^2\rangle$					1	1	1	1	2	3	5		$k-1$	$2k-3$	$3k-5$	$5k-10$
⟨321⟩						1	2	2	4	8	16		$2k-4$	$4k-8$	$8k-18$	$16k-40$
$\langle 31^3\rangle$							1	0	1	4	10		$k-3$	$k-3$	$4k-12$	$10k-30$
$\langle 2^3\rangle$								1	1	2	5			$k-2$	$2k-5$	$5k-15$
$\langle 2^21^2\rangle$									1	3	9			$k-3$	$3k-9$	$9k-30$
$\langle 21^4\rangle$										1	5				$k-4$	$5k-20$
$\langle 1^6\rangle$											1					$k-5$

	(2)	(1^2)	(0)
⟨6⟩	$\frac{1}{2}(k^2+k)$	$\frac{1}{2}(k^2+k)$	$\frac{1}{6}(k^3+3k^2+2k)$
⟨51⟩	$\frac{1}{2}(4k^2-4k)$	$\frac{1}{2}(5k^2-5k)$	$\frac{1}{6}(5k^3-5k)$
⟨42⟩	$\frac{1}{2}(6k^2-10k+4)$	$\frac{1}{2}(9k^2-19k+10)$	$\frac{1}{6}(9k^3-12k^2+3k)$
$\langle 41^2\rangle$	$\frac{1}{2}(6k^2-16k+8)$	$\frac{1}{2}(10k^2-28k+16)$	$\frac{1}{6}(10k^3-24k^2+8k)$
$\langle 3^2\rangle$	$\frac{1}{2}(3k^2-7k+4)$	$\frac{1}{2}(5k^2-13k+10)$	$\frac{1}{6}(5k^3-9k^2+4k)$
⟨321⟩	$\frac{1}{2}(8k^2-24k+16)$	$\frac{1}{2}(16k^2-56k+48)$	$\frac{1}{6}(16k^3-48k^2+32k)$
$\langle 31^3\rangle$	$\frac{1}{2}(4k^2-16k+12)$	$\frac{1}{2}(10k^2-44k+42)$	$\frac{1}{6}(10k^3-42k^2+38k-6)$
$\langle 2^3\rangle$	$\frac{1}{2}(2k^2-6k+4)$	$\frac{1}{2}(5k^2-21k+22)$	$\frac{1}{6}(5k^3-18k^2+19k-6)$
$\langle 2^21^2\rangle$	$\frac{1}{2}(3k^2-13k+12)$	$\frac{1}{2}(9k^2-43k+48)$	$\frac{1}{6}(9k^3-39k^2+36k)$
$\langle 21^4\rangle$	$\frac{1}{2}(k^2-5k+4)$	$\frac{1}{2}(5k^2-29k+36)$	$\frac{1}{6}(5k^3-27k^2+28k)$
$\langle 1^6\rangle$		$\frac{1}{2}(k^2-7k+10)$	$\frac{1}{6}(k^3-6k^2+5k)$

Table 6. Weight multiplicities $M^{[\lambda]}_{(\sigma)}$ of $O(2k)$.

$[\lambda]$ \ (σ)	(0)
[0]	1

	(1)
[1]	1

	(2)	(1^2)	(0)
[2]	1	1	$k-1$
$[1^2]$		1	k

	(3)	(21)	(1^3)	(1)
[3]	1	1	1	$k-1$
[21]		1	2	$2k-2$
$[1^3]$			1	$k-1$

	(4)	(31)	(2^2)	(21^2)	(1^4)	(2)	(1^2)	(0)
[4]	1	1	1	1	1	$k-1$	$k-1$	$\frac{1}{2}(k^2-k)$
[31]		1	1	2	3	$2k-2$	$3k-4$	$\frac{1}{2}(3k^2-5k+2)$
$[2^2]$			1	1	2	$k-2$	$2k-3$	$\frac{1}{2}(2k^2-2k)$
$[21^2]$				1	3	$k-1$	$3k-5$	$\frac{1}{2}(3k^2-5k)$
$[1^4]$					1		$k-2$	$\frac{1}{2}(k^2-k)$

	(5)	(41)	(32)	(31^2)	(2^21)	(21^3)	(1^5)	(3)	(21)	(1^3)	(1)
[5]	1	1	1	1	1	1	1	$k-1$	$k-1$	$k-1$	$\frac{1}{2}(k^2-k)$
[41]		1	1	2	2	3	4	$2k-2$	$3k-4$	$4k-6$	$\frac{1}{2}(4k^2-8k+4)$
[32]			1	1	2	3	5	$k-2$	$3k-5$	$5k-9$	$\frac{1}{2}(5k^2-11k+6)$
$[31^2]$				1	1	3	6	$k-1$	$3k-5$	$6k-12$	$\frac{1}{2}(6k^2-16k+10)$
$[2^21]$					1	2	5		$2k-4$	$5k-11$	$\frac{1}{2}(5k^2-13k+6)$
$[21^3]$						1	4		$k-2$	$4k-10$	$\frac{1}{2}(4k^2-12k+8)$
$[1^5]$							1			$k-3$	$\frac{1}{2}(k^2-3k+2)$

Table 6—continued.

	(6)	(51)	(42)	(41^2)	(3^2)	(321)	(31^3)	(2^3)	(2^21^2)	(21^4)	(1^6)	(4)	(31)	(2^2)	(21^2)	(1^4)
[6]	1	1	1	1	1	1	1	1	1	1	1	$k-1$	$k-1$	$k-1$	$k-1$	$k-1$
[51]		1	1	2	1	2	3	2	3	4	5	$2k-2$	$3k-4$	$3k-4$	$4k-6$	$5k-8$
[42]			1	1	1	2	3	3	4	6	9	$k-2$	$3k-5$	$4k-7$	$6k-11$	$9k-18$
$[41^2]$				1	0	1	3	1	3	6	10	$k-1$	$3k-5$	$3k-5$	$6k-12$	$10k-22$
$[3^2]$					1	1	1	1	2	3	5		$k-2$	$2k-3$	$3k-6$	$5k-11$
[321]						1	2	2	4	8	16		$2k-4$	$4k-8$	$8k-18$	$16k-40$
$[31^3]$							1	0	1	4	10		$k-2$	$k-2$	$4k-10$	$10k-28$
$[2^3]$								1	1	2	5			$k-3$	$2k-5$	$5k-14$
$[2^21^2]$									1	3	9			$k-2$	$3k-8$	$9k-27$
$[21^4]$										1	5				$k-3$	$5k-17$
$[1^6]$											1					$k-4$

	(2)	(1^2)	(0)
[6]	$\frac{1}{2}(k^2-k)$	$\frac{1}{2}(k^2-k)$	$\frac{1}{6}(k^3-k)$
[51]	$\frac{1}{2}(4k^2-8k+4)$	$\frac{1}{2}(5k^2-11k+6)$	$\frac{1}{6}(5k^3-9k^2+4k)$
[42]	$\frac{1}{2}(6k^2-14k+8)$	$\frac{1}{2}(9k^2-25k+18)$	$\frac{1}{6}(9k^3-21k^2+12k)$
$[41^2]$	$\frac{1}{2}(6k^2-16k+10)$	$\frac{1}{2}(10k^2-32k+26)$	$\frac{1}{6}(10k^3-30k^2+26k-6)$
$[3^2]$	$\frac{1}{2}(3k^2-9k+6)$	$\frac{1}{2}(5k^2-15k+12)$	$\frac{1}{6}(5k^3-12k^2+13k-6)$
[321]	$\frac{1}{2}(8k^2-24k+16)$	$\frac{1}{2}(16k^2-56k+48)$	$\frac{1}{6}(16k^3-48k^2+32k)$
$[31^3]$	$\frac{1}{2}(4k^2-12k+8)$	$\frac{1}{2}(10k^2-40k+40)$	$\frac{1}{6}(10k^3-36k^2+32k)$
$[2^3]$	$\frac{1}{2}(2k^2-6k+4)$	$\frac{1}{2}(5k^2-19k+16)$	$\frac{1}{6}(5k^3-15k^2+4k)$
$[2^21^2]$	$\frac{1}{2}(3k^2-11k+8)$	$\frac{1}{2}(9k^2-37k+36)$	$\frac{1}{6}(9k^3-30k^2+21k)$
$[21^4]$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{2}(5k^2-23k+26)$	$\frac{1}{6}(5k^3-18k^2+13k)$
$[1^6]$		$\frac{1}{2}(k^2-5k+6)$	$\frac{1}{6}(k^3-3k^2+2k)$

Table 7. Weight multiplicities $M_{(\Delta;\sigma)}^{[\Delta;\lambda]}$ of $O(2k)$.

$[\Delta;\lambda]$ \ $(\Delta;\sigma)$	$(\Delta;0)$
$[\Delta;0]$	1

	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;1]$	1	$k-1$

	$(\Delta;2)$	$(\Delta;1^2)$	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;2]$	1	1	$k-1$	$\frac{1}{2}(k^2-k)$
$[\Delta;1^2]$		1	$k-2$	$\frac{1}{2}(k^2-k)$

	$(\Delta;3)$	$(\Delta;21)$	$(\Delta;1^3)$	$(\Delta;2)$	$(\Delta;1^2)$	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;3]$	1	1	1	$k-1$	$k-1$	$\frac{1}{2}(k^2-k)$	$\frac{1}{6}(k^3-k)$
$[\Delta;21]$		1	2	$k-2$	$2k-4$	$\frac{1}{2}(2k^2-4k+2)$	$\frac{1}{6}(2k^3-8k+6)$
$[\Delta;1^3]$			1		$k-3$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{6}(k^3-7k)$

	$(\Delta;4)$	$(\Delta;31)$	$(\Delta;2^2)$	$(\Delta;21^2)$	$(\Delta;1^4)$	$(\Delta;3)$	$(\Delta;21)$	$(\Delta;1^3)$	$(\Delta;2)$	$(\Delta;1^2)$	$(\Delta;1)$	$(\Delta;0)$
$[\Delta;4]$	1	1	1	1	1	$k-1$	$k-1$	$k-1$	$\frac{1}{2}(k^2-k)$	$\frac{1}{2}(k^2-k)$	$\frac{1}{6}(k^3-k)$	$\frac{1}{24}(k^4+2k^3-k^2-2k)$
$[\Delta;31]$		1	1	2	3	$k-2$	$2k-4$	$3k-6$	$\frac{1}{2}(2k^2-4k+2)$	$\frac{1}{2}(3k^2-7k+4)$	$\frac{1}{6}(3k^3-3k^2-6k+6)$	$\frac{1}{24}(3k^4+6k^3-27k^2+18k)$
$[\Delta;2^2]$			1	1	2		$k-2$	$2k-5$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{2}(2k^2-6k+6)$	$\frac{1}{6}(2k^3-3k^2-5k+6)$	$\frac{1}{24}(2k^4+4k^3-26k^2+44k-24)$
$[\Delta;21^2]$				1	3		$k-3$	$3k-9$	$\frac{1}{2}(k^2-3k+2)$	$\frac{1}{2}(3k^2-11k+10)$	$\frac{1}{6}(3k^3-6k^2-9k+12)$	$\frac{1}{24}(3k^4+6k^3-51k^2+42k)$
$[\Delta;1^4]$					1			$k-4$		$\frac{1}{2}(k^2-5k+6)$	$\frac{1}{6}(k^3-3k^2-4k+6)$	$\frac{1}{24}(k^4+2k^3-25k^2+22k)$

where $\epsilon^{-\frac{1}{2}}$ corresponds to the representation of U(k) in which each group element A is mapped onto $(\det A)^{-\frac{1}{2}}$. For the group element (2.2), this gives rise to the character $\exp[-\frac{1}{2}\mathrm{i}(\phi_1+\phi_2+\ldots+\phi_k)]$. It should be stressed that these restrictions (4.7), (4.8), (6.4)–(6.6) are precisely the restrictions appropriate to classes of elements labelled by $\boldsymbol{\phi}$ in both groups and subgroup.

To exploit (6.4) it is only necessary to note that:

$$\mathrm{O}(2k) \qquad M_{(\sigma)}^{[\lambda]}=\sum_{\zeta,\beta} M_{(\bar{0};\,\sigma)}^{\{\zeta;\lambda/\zeta\beta\}}. \tag{6.7}$$

It follows that, for example

$$\begin{aligned}\mathrm{O}(2k) \qquad M_{(2)}^{[31]}&=\sum_{\zeta,\beta} M_{(\bar{0};\,2)}^{\{\bar{\zeta};31/\zeta\beta\}}=\sum_{\zeta}\left(M_{(\bar{0};\,2)}^{\{\bar{\zeta};31/\zeta\}}+M_{(\bar{0};\,2)}^{\{\bar{\zeta};\,2/\zeta\}}\right)\\ &=M_{(\bar{0};\,2)}^{\{\bar{1};\,3\}}+M_{(\bar{0};\,2)}^{\{\bar{1};\,21\}}+M_{(\bar{0};\,2)}^{\{0;\,2\}}=(k-1)+(k-2)+1=2k-2\end{aligned} \tag{6.8}$$

as required for agreement with the result (5.31).

This technique clearly provides a useful check on the internal consistency of the tabulated results.

Finally it should be remarked that although all weights and their multiplicities may be calculated using the character formulae given here, these formulae do not of course provide sufficient information to label uniquely all the basis states of the corresponding representations. This is a direct result of the formulae (5.12)–(5.17) containing negative as well as positive terms.

In order to label the basis states it is necessary to use instead of the character formulae the branching rules appropriate to the reduction from G to the toroidal subgroups T_G defined by (2.2)–(2.5). This work has been initiated elsewhere (King 1976) and leads to generalizations of both Gel'fand patterns and Young tableaux, as well as to the confirmation of the general formulae (5.11)–(5.17) for multiplicities. These generalizations of the Young tableaux provide a method of calculating the multiplicities involving only positive contributions, which again may be used to check the results tabulated here.

Turning to the tables themselves they represent the generalization to all the classical Lie groups of the results appropriate to U(k) tabulated by Blaha (1969). For convenience the tables have been presented in terms of rectangular blocks but it is clear that with a suitable ordering of these blocks the multiplicity matrix of each classical Lie group G is upper triangular with each diagonal element unity. This implies the invertibility of these matrices. This has been illustrated in the case of covariant tensor representation of U(k) by Blaha (1969) who calculated the coefficient B_{λ}^{μ} appearing in the identity:

$$k_{\mu}(x_1, x_2, \ldots, x_n)=\sum_{\lambda} B_{\lambda}^{\mu} e_{\lambda}(x_1, x_2, \ldots, x_n) \tag{6.9}$$

which is the inverse of (3.4) mentioned in § 3. The other inverse multiplicity matrices may be calculated in a straightforward way. They are of use in determining, for example, plethysms associated with the orthogonal and symplectic groups. These define branching rules appropriate to certain group–subgroup restrictions (Plunkett 1972).

The weight multiplicities of the covariant tensor irreducible representations of U(k) are independent of the rank k. In general all the other weight multiplicities are dependent on k. The powers of k which appear in any block of the multiplicity matrix are determined by the differences, $l-s$ and $m-t$, between the numbers of which $\boldsymbol{\lambda}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$, $\boldsymbol{\tau}$ are partitions. The great merit of the results tabulated here is that they apply to groups of arbitrarily high rank and to unitary representations of arbitrarily high dimensions. Thus unlike the methods reviewed by Beck and Kolman (1972b), Gruber (1973) and Kolman and Beck (1973b) the method presented here does not involve carrying out essentially different calculations for groups differing only in rank.

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